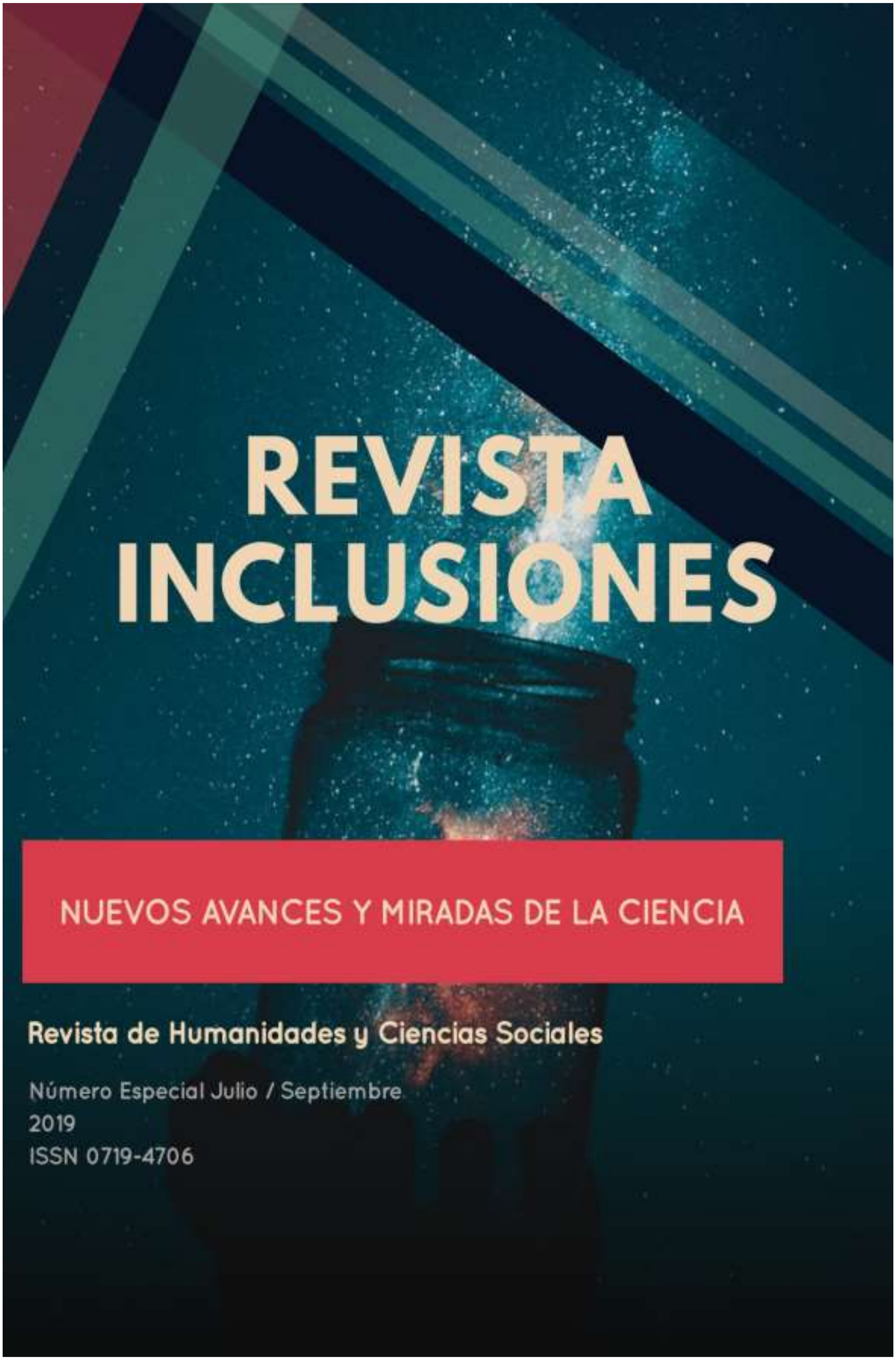
REVISTA INCLUSIONES
NUEVOS AVANCES Y MIRADAS DE LA CIENCIA
Revista de Humanidades y Ciencias Sociales
Número Especial Julio / Septiembre
2019
ISSN 0719-4706

REVISTA INCLUSIONES
REVISTA DE HUMANIDADES Y CIENCIAS SOCIALES

CUADERNOS DE SOFÍA EDITORIAL

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Revista Inclusiones, se encuentra indizada en:

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REVISTA INCLUSIONES
REVISTA DE HUMANIDADES Y CIENCIAS SOCIALES

CUADERNOS DE SOFÍA EDITORIAL

ISSN 0719-4706 - Volumen 6 / Número Especial / Julio – Septiembre 2019 pp. 165-177

SOLUTION OF EQUATION WITH FRIEDLY INTEGRALS BY COLLOCATION METHOD

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Fecha de Recepción: 19 de marzo de 2019 – **Fecha Revisión**: 28 de abril de 2019

Fecha de Aceptación: 21 de junio de 2019 – **Fecha de Publicación**: 01 de julio de 2019

Abstract

This article describes a collocation method for solving an equation containing the fractional Riemann-Liouville integral of the form, form, где – regular integral, – known constant coefficients, – given as well – the desired function. An approximate solution is sought in the form of an algebraic polynomial. In constructing the computational scheme of the method, quadrature interpolation type formulas are used to calculate the fractional Riemann – Liouville integral, which were constructed in previous works of the authors. According to the collocation method, unknown coefficients are found by solving a system of linear equations of the collocation method. The substantiation of the constructed method is carried out, which implies the proof of the existence, the uniqueness of the approximate solution, and also its stability. A model equation was constructed and its numerical solution was performed by the indicated collocation method. The implementation of the method was carried out in the Wolfram Mathematica system.

Keywords

Fractional integral – Fractional derivative – Riemann-Liouville integral
Fractional integration – Fractional differentiation

Para Citar este Artículo:

Galimyanov, Anis F.; Gilemzyanov, Almaz F. y Minnegalieva, Chulpan B. Solution of equation with friendly integrals by collocation method. Revista Inclusiones Vol: 6 num Esp Jul-Sep (2019): 165-177.

Introduction

Fractional calculation methods are widely used in science and technology. For example, with the help of fractional derivatives, the propagation of signals and the behavior of biological systems are investigated[1]. Therefore, the study and conclusion of new methods of fractional calculus is the current direction. At present, fractional integrals and derivatives of Riemann-Liouville, Weil, Hadamard, etc. are used and investigated. Previously, we derived formulas for representing an interpolation operator that assigns 2π - a periodic function with zero mean value, a trigonometric polynomial, and a quadrature formula for the fractional Weyl integral is constructed. Weil fractional integrals are also studied in Wang, et al.[2], the properties of fractional derivatives of Riemann-Liouville and Hadamard were generalized, it was shown that, under certain assumptions, different definitions of fractional derivatives and integrals can be considered as special cases of a single unifying model of fractional calculus.

Currently, active research is being conducted on approximate methods for solving integral and differential equations. For example, fractional differential equations are considered in Vu[3]. In Malawi, et al.[4], approximate calculations of equations with fractional derivatives are considered. This makes the task of finding and substantiating methods for solving integral equations containing fractional integrals relevant.

If a $\varphi(x) \in C[a,b]$ integrals $(I_{a+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\,dt, x > a$, and $(I_{b-}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t)}{(x-t)^{1-\alpha}}\,dt, x < b$, where $\alpha > 0$, are called fractional integrals α in Riemann-Liouville. Integrals of this type are studied by mathematicians. For example, inequalities were proved in Nasibullin[5] in the case when the weight functions have power and logarithmic singularities. Earlier we constructed quadrature interpolation type formulas for calculating the fractional Riemann – Liouville integral $(I_{a+}^{\alpha}\varphi)(x)$ and the quadrature formula for the singular integral with the Cauchy kernel of the fractional Riemann – Liouville integral.

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Methods

The computational scheme of the collocation method is constructed using the previously derived quadrature formulas. In assessing the accuracy of finding an approximate solution, the properties of the norm in linear spaces and the properties of the Lagrange operator are used. To verify the stated method, calculations are carried out in the computer algebra system Wolfram Mathematica.

Results and Discussion

Consider an equation of the form $A\varphi(x)+B(I_{a+}^{\alpha}\varphi)(x)+C(T\varphi)(x)=f(x)$ (1),

Where $(T\varphi)(x)=\frac{1}{\pi}\int_a^b h(x,t)\varphi(t)dt$ – regular integral, A,B,C – known constant coefficients $f(t),h(x,t)$ – given, a $\varphi(x)$ – desired function, $x\in[a,b]$

This equation is the Fredholm equation of the second kind, therefore the problem of solving such an equation is correctly posed. We will solve this equation with the collocation method.

Denote by $C[a,b]$ continuous space on $[a,b]$ functions. Its rate is determined by the formula $\|\varphi\|_C=\max_{a\le x\le b}|\varphi(x)|$.

We write the computational scheme of the collocation method. An approximate solution will be sought in the form of an algebraic polynomial

$$\varphi_n(x)=\sum_{k=0}^{n}a_k(x-a)^k \quad (2),$$

Where are the unknowns a_k we define from the condition of equality to zero residuals in the nodes $x_j=\cos\frac{2j+1}{2n+2}\pi$ $j=\overline{0,n}$?

We substitute this polynomial into the equation and transform it, considering each term separately, it was shown that

$$(I_{a+}^{\alpha}\varphi_n)(x)=\sum_{k=0}^{n}a_k(x-a)^{k+\alpha}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}.$$

Consider a special case when $a=-1$ $b=1$.
Then we have

$$(I_{-1+}^{\alpha}\varphi_n)(x)=\sum_{k=0}^{n}a_k(x+1)^{k+\alpha}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}$$

Consider the last term.

$$(T\varphi_n)(x)=\frac{1}{\pi}\int_{-1}^{1}h(x,t)\varphi_n(t)dt=\frac{1}{\pi}\int_{-1}^{1}h(x,t)\sum_{k=0}^{n}a_k(t+1)^k dt=$$
$$\frac{1}{\pi}\sum_{k=0}^{n}a_k\int_{-1}^{1}h(x,t)(t+1)^k dt.$$

As a result, we obtain that the equation

$$A\varphi_n(x)+B(I_{a+}^{\alpha}\varphi_n)(x)+C(T\varphi_n)(x)\approx f(x)$$

matches the equation

$$A\varphi_n(x)+B\sum_{k=0}^{n}a_k(x+1)^{k+\alpha}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}+\frac{C}{\pi}\sum_{k=0}^{n}a_k\int_{-1}^{1}h(x,t)(t+1)^k dt\approx f(x).$$

Substitute the nodes $x_j=\cos\frac{2j+1}{2n+2}\pi$ $j=\overline{0,n}$, and get the collocation method system:

$$A\varphi_n(x_j)+B\sum_{k=0}^{n}a_k(x_j+1)^{k+\alpha}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}+\frac{C}{\pi}\sum_{k=0}^{n}a_k\int_{-1}^{1}h(x_j,t)(t+1)^k dt\approx f(x_j)$$, (3) $j=\overline{0,n}$.

We present a theorem that we will use later when estimating the rate of convergence of the exact and approximate solutions of the system.

Theorem 1. *Let the following conditions be satisfied:*

1) operator $A:X\to Y$ *it has* A^{-1} *– limited;*
2) $\varepsilon_n=\|A-A_n\|_{X_n\to Y}\to 0$ $n\to\infty$ *,;*
3) $\dim X_n=\dim Y_n=N(n)$.

Then starting with some such that $q_n=\|A^{-1}\|\varepsilon_n<1$, *the equation* $A_nx_n=y_n$ $x_n\in X_n\subset X$, $y_n\in Y_n\subset Y$, *(где* X,Y *– linear normalized spaces,* $X_n\subset X$ *и* $Y_n\subset Y$ *– their arbitrary fixed finite-dimensional subspaces of the same dimension)* *are uniquely solvable, and* $\|x_n^*\|\le\|A_n^{-1}\|\|y_n\|$ $\|A_n^{-1}\|\le\frac{\|A^{-1}\|}{1-q_n}\le 2\|A^{-1}\|$, *If, moreover, is satisfied*

4) $\delta_n = \|y - y_n\|_Y \to 0$ $n \to \infty$*, that approximate solution* $x_n^* = A_n^{-1} y_n$ *converges to the exact* $x^* = A^{-1} y \in X$ *according to the norm* X *, at the same time, the error of the approximate solution can be estimated by the inequality*

$$\|x^* - x_n^*\| \le \frac{\|A^{-1}\|}{1 - q_n}\left(\|y - y_n\| + q_n\|y\|\right).$$

Based on a variant of the general theory of approximate methods of L. V. Kantarovich proposed by B. G. Gabdulkhaev, we will prove the following theorem:

Theorem 2. *Let the following conditions be satisfied:*

1) the homogeneous equation corresponding to equation (1) has only a trivial solution;

2) the right side of the equation (1) $f(x)$ *belongs* $C[-1,1]$*;*

3) core belongs $C[-1,1] \times C[-1,1]$*;*

4) collocation nodes are $x_j = \cos\frac{2j+1}{2n+2}\pi$ $j = \overline{0, n}$*, – Chebyshev nodes of the 1st kind.*

Then the system of collocation method (3) has the only solution $\{a_k^*\}_0^n$ *for* $\forall n \ge n_0$.

The approximate solution $\varphi_n^*(x) = \sum_{k=0}^{n} a_k^*(x+1)^k$ *converges evenly to the exact solution* $\varphi^*(x)$ *the equations (1) in terms of space* $C[-1,1]$*, the rate of convergence is determined by the following ordinal ratio:*

$$\|\varphi^* - \varphi_n^*\|_C = O\left\{(1 + \ln n)\left(E_n(f)_C + E_n^x(h)_{C\times C} + \frac{1}{n^\alpha}\right)\right\}$$ *at* $n \to \infty$,

Where $E_n(f)_C$ *u* $E_n^x(h)_{C\times C}$ *– best uniform element approximations* f *and* h *on variable view elements* f_n h_n^x *and respectively.*

Evidence. We introduce the main space $\Phi = F = C[-1,1]$ with the norm $\|\varphi\|_C = \max_{-1\le x\le 1}|\varphi(x)|$.

Choose its subspace $\Phi_n = H_{n-1} \cap \Phi$, where H_m – the set of all algebraic polynomials of degree not higher $m(m \in N)$.

Equation (1) can be written as:

$K\varphi \equiv A\varphi + BI^{\alpha}_{-1+}\varphi + CT\varphi = f$, $\varphi, f \in \Phi$.

It can be noted that conditions 1, 3 imply that the operator $T : C[-1,1] \to C[-1,1]$ is continuous. Then, by the Fredholm theory, we obtain that $\exists K^{-1} : \Phi \to \Phi$ – limited. System (3) has the form $P_n K\varphi_n = K_n\varphi_n \equiv P_n(A\varphi_n + BI^{\alpha}_{-1+}\varphi_n + CT\varphi_n) = P_n f$.

Take as an operator P_n Lagrange operator $P_n = L_n$ degrees n, built on collocation nodes 4), which associates any function with its algebraic interpolation Lagrange polynomial.

This operator has the following properties: $P_n^2 = P_n$ that is, it is projection.

It is also known that $\|P_n\|_{C\to C} = O(\ln n)$ $n \to \infty$ at.

We will consider $\delta_n = \|f - P_n f\|_\Phi$. Add and subtract the best approximation element Q_n for function f and break into two norms. Then we get
$\delta_n \le \|f - Q_n\|_\Phi + \|Q_n - P_n f\|_\Phi$.

We use the property of the Lagrange operator that it leaves the polynomial of the same degree unchanged.
$\delta_n \le \|f - Q_n\|_\Phi + \|P_n Q_n - P_n f\|_\Phi$.

From the second term, we take the operator norm P_n and note that
$\|f - Q_n\|_\Phi = E_n(f)_\Phi$.

In this way, $\delta_n \le \|f - Q_n\|_\Phi + \|P_n\|_{\Phi\to\Phi}\|Q_n - f\|_\Phi = \left(1 + \|P_n\|_{\Phi\to\Phi}\right)\|f - Q_n\|_\Phi =$
$= O\{(1 + \ln n)E_n(f)_\Phi\} \to 0$, at $n \to \infty$.

Now take $\forall\varphi_n \in \Phi_n$ and consider the value

$\|K\varphi_n - K_n\varphi_n\|_\Phi = \|A\varphi_n + BI^{\alpha}_{-1+}\varphi_n + CT\varphi_n - P_n\left(A\varphi_n + BI^{\alpha}_{-1+}\varphi_n + CT\varphi_n\right)\|_\Phi$.

We divide the norm into terms

$$\|K\varphi_n - K_n\varphi_n\|_\Phi \le A\|\varphi_n - P_n\varphi_n\|_\Phi + B\|I^\alpha_{-1+}\varphi_n - P_n I^\alpha_{-1+}\varphi_n\|_\Phi + C\|T\varphi_n - P_n T\varphi_n\|_\Phi.$$

Consider each addend separately.

It's obvious that $\|\varphi_n - P_n\varphi_n\|_\Phi = \|\varphi_n - \varphi_n\| = 0$ since the Lagrange, the operator has the property that leaves the polynomial of the same degree unchanged.

Now consider $\|I^\alpha_{-1+}\varphi_n - P_n I^\alpha_{-1+}\varphi_n\|_\Phi$.

Add and subtract the best approximation element Q_n for function $I^\alpha_{-1+}\varphi_n$ and break into two norms. Then we get

$$\|I^\alpha_{-1+}\varphi_n - P_n I^\alpha_{-1+}\varphi_n\|_\Phi \le \|I^\alpha_{-1+}\varphi_n - Q_n\|_\Phi + \|Q_n - P_n I^\alpha_{-1+}\varphi_n\|_\Phi.$$

Again we use the property of the operator Lagrange

$\|I^\alpha_{-1+}\varphi_n - P_n I^\alpha_{-1+}\varphi_n\|_\Phi \le \|I^\alpha_{-1+}\varphi_n - Q_n\|_\Phi + \|P_n Q_n - P_n I^\alpha_{-1+}\varphi_n\|_\Phi$. From the second term, we take the operator norm P_n and note that $\|I^\alpha_{-1+}\varphi_n - Q_n\|_\Phi = E_n(I^\alpha_{-1+}\varphi_n)_\Phi$. As a result, we get $\|I^\alpha_{-1+}\varphi_n - P_n I^\alpha_{-1+}\varphi_n\|_\Phi \le \|I^\alpha_{-1+}\varphi_n - Q_n\|_\Phi + \|P_n\|_{\Phi\to\Phi}\|Q_n - I^\alpha_{-1+}\varphi_n\|_\Phi = = (1 + \|P_n\|_{\Phi\to\Phi})\|I^\alpha_{-1+}\varphi_n - Q_n\|_\Phi =$ $(1 + \|P_n\|_{\Phi\to\Phi})E_n(I^\alpha_{-1+}\varphi_n)_\Phi$.

Consider the value $E_n(I^\alpha_{-1+}\varphi_n)_\Phi$. We estimate this value using the following theorem.

D. Jackson's theorem. *If* E_n *there is the best approximation of the function* $f(x) \in C[a,b]$ *polynomials from* H_n*, that* $E_n \le 12\omega\left(\frac{b-a}{2n}\right)$*, где* $\omega(\delta)$ *– modulus of continuity* $f(x)$*.*

Thus, we get $E_n(I^\alpha_{-1+}\varphi_n)_\Phi \le 12\omega\left(\frac{1}{n}\right) = 12\omega\left(I^\alpha_{-1+}\varphi_n, \frac{1}{n}\right)$.

By definition, the modulus of continuity is defined by the following formula $\omega(f,\delta) = \sup\limits_{\substack{|x'-x''|\le\delta \\ x'\ne x''}} |f(x') - f(x'')|$ $\forall x', x'' \in C[a,b]$.

Take $-1 < x' < x''$. Then in our case, we have

$$\omega\left(I_{-1+}^{\alpha}\varphi_n, \frac{1}{n}\right) = \sup_{\substack{|x'-x''|\le\frac{1}{n} \\ -1<x'<x''}} \left|I_{-1+}^{\alpha}(\varphi_n, x') - I_{-1+}^{\alpha}(\varphi_n, x'')\right| =$$

$$= \sup_{\substack{|x'-x''|\le\frac{1}{n} \\ -1<x'<x''}} \left|\frac{1}{\Gamma(\alpha)}\int_{-1}^{x'}\frac{\varphi_n(t)}{(x'-t)^{1-\alpha}}dt - \frac{1}{\Gamma(\alpha)}\int_{-1}^{x''}\frac{\varphi_n(t)}{(x''-t)^{1-\alpha}}dt\right|.$$

In the second integral, we divide the integration interval into two intervals $-1 < t < x'$ $x' < t < x''$ и, we add the module under the integral sign, we use the property $\sup|a-b| \le \sup|a| + \sup|b|$ and get the following

$$\omega\left(I_{-1+}^{\alpha}\varphi_n, \frac{1}{n}\right) \le \frac{1}{\Gamma(\alpha)}\left(\sup_{|x'-x''|\le\frac{1}{n}}\int_{-1}^{x'}\left|\frac{\varphi_n(t)}{(x'-t)^{1-\alpha}} - \frac{\varphi_n(t)}{(x''-t)^{1-\alpha}}\right|dt + \sup_{|x'-x''|\le\frac{1}{n}}\int_{x'}^{x''}\left|\frac{\varphi_n(t)}{(x''-t)^{1-\alpha}}\right|dt\right).$$

Evaluate the function φ_n maximum and make it for the integral sign

$$\omega\left(I_{-1+}^{\alpha}\varphi_n, \frac{1}{n}\right) \le \frac{1}{\Gamma(\alpha)}\max_{-1\le x\le 1}|\varphi_n(x)| \sup_{|x'-x''|\le\frac{1}{n}}\int_{-1}^{x'}\left|\frac{1}{(x'-t)^{1-\alpha}} - \frac{1}{(x''-t)^{1-\alpha}}\right|dt +$$

$$+ \frac{1}{\Gamma(\alpha)}\max_{-1\le x\le 1}|\varphi_n(x)| \sup_{|x'-x''|\le\frac{1}{n}}\int_{x'}^{x''}\left|\frac{1}{(x''-t)^{1-\alpha}}\right|dt.$$

Integral expressions are positive, so you can remove the modules and calculate the remaining integrals

$$\omega\left(I_{-1+}^{\alpha}\varphi_n, \frac{1}{n}\right) \le \frac{1}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi} \sup_{|x'-x''|\le\frac{1}{n}}\left(\frac{(x'+1)^{\alpha}}{\alpha} - \frac{(x''+1)^{\alpha}}{\alpha} + \frac{(x''-x')^{\alpha}}{\alpha}\right) +$$

$$+ \frac{1}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi} \sup_{|x'-x''|\le\frac{1}{n}}\frac{(x''-x')^{\alpha}}{\alpha} =$$

$$= \frac{1}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi} \sup_{|x'-x''|\le\frac{1}{n}}\left(\frac{(x''-x')^{\alpha}}{\alpha} - \left(\frac{(x''+1)^{\alpha}}{\alpha} - \frac{(x'+1)^{\alpha}}{\alpha}\right)\right) +$$

$$+ \frac{1}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi} \sup_{|x'-x''|\le\frac{1}{n}}\frac{(x''-x')^{\alpha}}{\alpha}.$$

The magnitude $\frac{(x''-x')^{\alpha}}{\alpha}-\left(\frac{(x''+1)^{\alpha}}{\alpha}-\frac{(x'+1)^{\alpha}}{\alpha}\right)$ is positive. Since if the integrand is positive, then the integral of it will be positive. By assumption $-1<x'<x''$ magnitudes $\frac{(x''-x')^{\alpha}}{\alpha}$ $\frac{(x''+1)^{\alpha}}{\alpha}-\frac{(x'+1)^{\alpha}}{\alpha}$ and are also positive. Therefore, we can discard the last term. We get the following

$$\omega\left(I_{-1+}^{\alpha}\varphi_n,\frac{1}{n}\right)\le\frac{1}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi}\sup_{|x'-x''|\le\frac{1}{n}}\left(\frac{(x''-x')^{\alpha}}{\alpha}-\left(\frac{(x''+1)^{\alpha}}{\alpha}-\frac{(x'+1)^{\alpha}}{\alpha}\right)\right)+$$

$$+\frac{1}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi}\sup_{|x'-x''|\le\frac{1}{n}}\frac{(x''-x')^{\alpha}}{\alpha}\le\frac{1}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi}\sup_{|x'-x''|\le\frac{1}{n}}\frac{(x''-x')^{\alpha}}{\alpha}+$$

$$+\frac{1}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi}\sup_{|x'-x''|\le\frac{1}{n}}\frac{(x''-x')^{\alpha}}{\alpha}=\frac{2}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi}\sup_{|x'-x''|\le\frac{1}{n}}\frac{(x''-x')^{\alpha}}{\alpha}.$$

This, in turn, can be estimated as

$$\omega\left(I_{-1+}^{\alpha}\varphi_n,\frac{1}{n}\right)\le\frac{2}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi}\sup_{|x'-x''|\le\frac{1}{n}}\frac{(x''-x')^{\alpha}}{\alpha}\le\frac{2}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi}\frac{1}{\alpha n^{\alpha}}.$$

Therefore, we obtain that

$$E_n\left(I_{-1+}^{\alpha}\varphi_n\right)_{\Phi}\le 12\omega\left(I_{-1+}^{\alpha}\varphi_n,\frac{1}{n}\right)=\frac{24}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi}\frac{1}{\alpha n^{\alpha}}.$$

Consequently $\left\|I_{-1+}^{\alpha}\varphi_n-P_nI_{-1+}^{\alpha}\varphi_n\right\|_{\Phi}\le\left(1+\|P_n\|_{\Phi\to\Phi}\right)E_n\left(I_{-1+}^{\alpha}\varphi_n\right)_{\Phi}\le\left(1+\|P_n\|_{\Phi\to\Phi}\right)\frac{24}{\Gamma(\alpha)}\|\varphi_n\|_{\Phi}\frac{1}{\alpha n^{\alpha}}$.

It remains to consider the last term $\|T\varphi_n-P_nT\varphi_n\|_{\Phi}=\|Th\varphi_n-P_nTh\varphi_n\|_{\Phi}=\|T(h-P_n^xh)\varphi_n\|_{\Phi}\le\|h-P_n^xh\|_{\Phi\times\Phi}\|\varphi_n\|_{\Phi}$.

Add and subtract the best approximation element Q_n for function h, we divide the norm into two components, again we use the property of the Lagrange operator that it leaves the same degree of the polynomial without change from the second term, we derive the operator norm P_n and note that $\|h-Q_n\|_{\Phi}=E_n^x(h)_{\Phi\times\Phi}$. As a result, we get

$$\|T\varphi_n-P_nT\varphi_n\|_{\Phi}\le\left(\|h-Q_n\|_{\Phi\times\Phi}+\|P_n\|_{\Phi\to\Phi}\|Q_n-h\|_{\Phi\times\Phi}\right)\cdot\|\varphi_n\|_{\Phi}=$$

$$= \left(1 + \|P_n\|_{\Phi \to \Phi}\right) E_n^x(h)_{\Phi\times\Phi} \|\varphi_n\|_\Phi .$$

Thus, we have that $\forall \varphi_n \in \Phi_n$ inequality holds

$$\|K\varphi_n - K_n\varphi_n\|_\Phi \le \left(1 + \|P_n\|_{\Phi\to\Phi}\right)\left(\frac{24}{\Gamma(\alpha)}\frac{1}{\alpha n^\alpha} + E_n^x(h)_{\Phi\times\Phi}\right)\cdot\|\varphi_n\|_\Phi .$$

As known, $\|P_n\|_{C\to C} = O(\ln n)$ $n \to \infty$ at, then we get the following ordinal ratio

$$\|K\varphi_n - K_n\varphi_n\|_\Phi = O\left(\left(1 + \ln n\right)\left(\frac{1}{n^\alpha} + E_n^x(h)_{\Phi\times\Phi}\right)\right)\cdot\|\varphi_n\|_\Phi .$$

Magnitude $\|K\varphi_n - K_n\varphi_n\|_\Phi$ can be painted as $\|K\varphi_n - K_n\varphi_n\|_\Phi \le \|K - K_n\|_{\Phi_n\to\Phi}\|\varphi_n\|_\Phi$ it follows that

$$\varepsilon_n = \|K - K_n\|_{\Phi_n\to\Phi} = O\left(\left(1 + \ln n\right)\left(\frac{1}{n^\alpha} + E_n^x(h)_{\Phi\times\Phi}\right)\right) \to 0 \text{ at } n \to \infty .$$

As a result, we are in the conditions of Theorem 1, we find that the system of the collocation method (3) has a unique solution $\{a_k^*\}_0^n$ for $\forall n \ge n_0$ an approximate solution $\varphi_n^*(x)$ of the form (2) converges uniformly to the exact solution $\varphi^*(x)$ equations (1) in the norm of space $C[-1,1]$, the rate of convergence is determined by the following ordinal ratio:

$$\|\varphi^* - \varphi_n^*\|_C = O\{\varepsilon_n + \delta_n\} = O\left\{(1 + \ln n)\left(E_n(f)_C + E_n^x(h)_{C\times C} + \frac{1}{n^\alpha}\right)\right\} \text{ at } n \to \infty .$$

under Theorem proved.

Wolfram Mathematica Calculations

To test the method described, we used the computer algebra system Wolfram Mathematica. This system is used in scientific calculations, for example.

To check the system of collocation method (3), take the function $\varphi(x) = (x-a)^{\beta-1}$, which is the exact solution of the system. $\varphi_n(x) = \sum_{k=0}^{n} a_k (x-a)^k$ – the function that we found in the form of an algebraic polynomial, the coefficients a_k are solutions of system (3).

Figure 1 plotted functions $\varphi(x)$ and $\varphi_n(x)$ for $\alpha = 0,5$, $\beta = 0,6$, $A = 2$, $B = 3$, $C = 4$ at $n = 5$

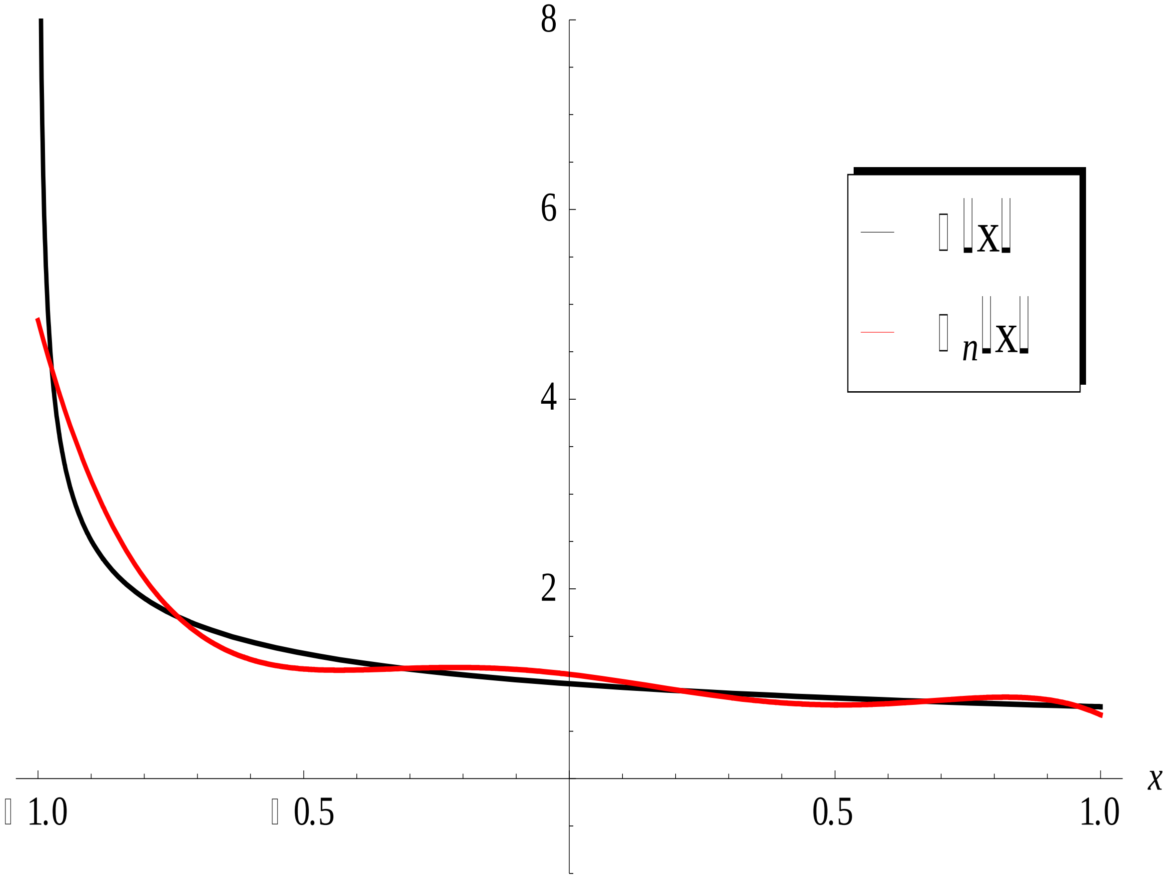

Figure 1

The table shows the values of the functions. $\varphi(x)$ and $\varphi_n(x)$:

x	-0,9	-0,7	-0,5	-0,3	-0,1	0,1	0,3	0,5	0,7	0,9
$\varphi(x)$	2,51189	1,61864	1,31951	1,15335	1,04304	0,96259	0,90039	0,85028	0,80876	0,77356
$\varphi_n(x)$	3,14621	1,53604	1,15642	1,16355	1,15178	1,02310	0,85649	0,77736	0,82696	0,83177

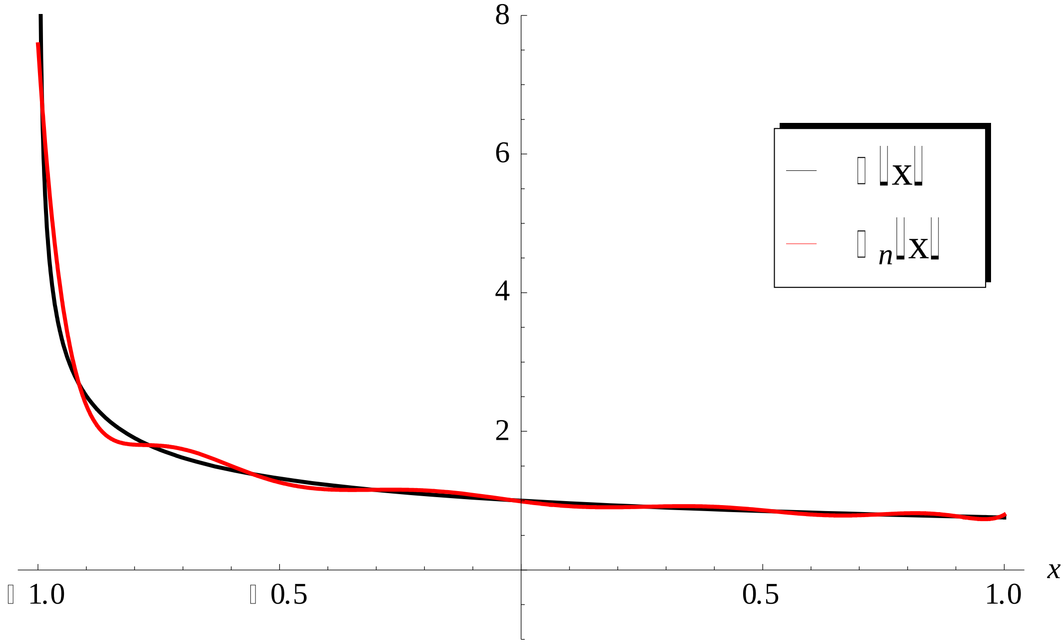

Fig. 2
plotted with $n=10$

The table shows the values of the functions. $\varphi(x)$ and $\varphi_n(x)$ at $n=10$:

x	-0,9	-0,7	-0,5	-0,3	-0,1	0,1	0,3	0,5	0,7	0,9
$\varphi(x)$	2,51189	1,61864	1,31951	1,15335	1,04304	0,96259	0,90039	0,85028	0,80876	0,77356
$\varphi_n(x)$	2,37658	1,74321	1,26319	1,15758	1,08413	0,92073	0,91900	0,86180	0,78954	0,77697

Summary

The stated collocation method allows solving equations of the form $A\varphi(x)+B(I_{a+}^{\alpha}\varphi)(x)+C(T\varphi)(x)=f(x)$. The solution is sought in the form of an algebraic polynomial. $\varphi_n(x)=\sum_{k=0}^{n}a_k(x-a)^k$ .. As shown by calculations in the Wolfram Mathematica system, with an increasing number of nodes, accuracy increases.

Conclusions

Thus, a collocation method is proposed for solving an equation of the form $A\varphi(x)+B(I_{a+}^{\alpha}\varphi)(x)+C(T\varphi)(x)=f(x)$, which is the Fredholm equation of the second kind. The method was verified using a function that is an exact solution to the equation. The necessary theorems are proved and formulas are derived for estimating the rate of convergence of an approximate solution of an equation to an exact solution. Calculations in the computer mathematics system have shown that when searching for a solution to an equation in the form of an algebraic polynomial, to achieve the required accuracy, it is necessary to increase the number of nodes $x_j = \cos\frac{2j+1}{2n+2}\pi$

Acknowledgments

The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University.

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